

The Golden Ratio Triple Lattice PETs

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Abstract

We study a one-parameter family of multigraph polytope exchange transformations which were introduced by R. Schwartz in 2013. With the help of computer, we will prove that there exists a renormalization scheme in a neighborhood of the golden ratio $\phi = (\sqrt{5} - 1)/2$ in the interval $(0, 1)$. We analyze the limit set Λ_ϕ with respect to the golden ratio ϕ . We show that Λ_ϕ has Lebesgue measure zero. Moreover, Λ_ϕ is the limit of embedded polygons in \mathbb{R}^2 and its Hausdorff dimension satisfies the inequality $1 < \dim_H(\Lambda_\phi) = \log(\sqrt{2} - 1)/\log(\phi) < 2$.

1 Introduction

1.1 Background

Polytope exchange transformations (PETs) are dynamical systems that generalize interval exchange transformations (IETs). The definition of PETs is given as follows:

Definition 1.1. Let X be a polytope. A *polytope exchange transformation (PET)* is determined by two partitions of small polytopes $\mathcal{A} = \{A_i\}_{i=1}^m$ and $\mathcal{B} = \{B_i\}_{i=1}^m$ of X . For each $A_i \in \mathcal{A}$, there exists a vector V_i satisfying the property that

$$B_i = A_i + V_i.$$

A PET $f : X \rightarrow X$ is defined by the formula:

$$f(x) = x + V_i, \quad \forall x \in \text{Int}(A_i).$$

We call V_i a translation vector of f on A_i . Note that the PET f is not defined on the boundary ∂A_i for each i .

The 1-dimensional examples of PETs are interval exchange transformations (IETs) which are well-studied for more than 40 years and come up in many areas of mathematics such as ergodic theory, billiards and Teichmüller theory, see [10] and [12] and [2] for surveys. The paper [7] introduces rectangles exchanges which are the products of IETs. The author provides a criterion for the minimal rectangle exchanges. For the work concerning entropy, a general result is proved in [5] and [6] that every PET has zero entropy.

The study of piecewise isometry is closely related to the study of PETs. The paper [4] gives a nice survey on piecewise isometries. Here is the definition: let X be a subset of \mathbb{R}^n and $\mathcal{P} = \{P_i\}_{i=1}^n$ be a partition on X . A *piecewise isometry* is a map $T : X \rightarrow X$ such

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that its restriction to each element P_i of the partition \mathcal{P} is a Euclidean isometry. There is a natural construction of PETs from piecewise rational rotation which is a particular case of the piecewise isometry in \mathbb{R}^2 . Let X be a polygon and \mathcal{P} be a partition on X where each element of \mathcal{P} is a polygon. A *piecewise rational rotation* on X is a map $T : X \rightarrow X$ such that the restriction $T|_{P_i}$ is either a translation or a rotation by rational multiple of π . There is a polygon exchange transformation $\tilde{T} : \tilde{X} \rightarrow \tilde{X}$ on the covering space (\tilde{X}, π) for $\pi : \tilde{X} \rightarrow X$ such that $\pi \circ \tilde{T} = T \circ \pi$.

The scheme of renormalization is used to understand PETs (or more generally, piecewise isometries) in a lot of cases. Renormalization is a tool to zoom into the space and accelerate the orbits of points along time. To see this, we provide some basic definitions:

Definition 1.2. Let Y be a subset of X . Given a map $f : X \rightarrow X$, the *first return* $f|_Y : Y \rightarrow Y$ is a map assigns every point $p \in Y$ to the first point in the forward orbit of p lies in Y under f , i.e.

$$f|_Y(p) = f^k(p) \quad \text{where } k = \min\{f^k(p) \in Y\} \quad k \geq 0.$$

As an example of renormalization, it might happen that the first return map $f|_Y$ of a PET $f : X \rightarrow X$ on a subset Y of X is affine conjugate to the original map f . It follows that the first return map $f|_Y$ is a PET on the subset Y . More importantly, the dynamical behavior of $f|_Y$ is similar to the dynamics of the map f . Therefore, once the renormalization scheme is discovered, we can explore more details of the PETs.

A classical example of renormalizable piecewise rational rotation is provided in the survey paper [4]. The paper [8] and [9] gives the first example of PETs in 2-dimensional parameter space which is invariant under renormalization. The renormalization scheme arises from Truchet tilings. A recent paper [1] studies an example of piecewise isometries which is very similar to the one in [8]. A renormalization scheme of the system is discovered.

In this paper, we will study an one-parameter family of PETs called the triple lattice PETs. The monograph [11] on octagonal PETs is very close to our work here. There is a local equivalence between octagonal PETs and outer billiards on semi-regular octagons. The family of octagonal PETs has a renormalization scheme in which the $(2, 4, \infty)$ hyperbolic reflection triangle group acts on the parameter space by linear fraction transformations.

1.2 Multigraph PETs

A general method of constructing PETs called multigraph PETs is described by Schwartz in [11]. We provide some basic definitions here:

Definition 1.3. A *multigraph* is a monogon-free graph such that two vertices may be connected by more than one edge.

Let V be a set of vertices labeled by convex polytopes and E be a set of edges labeled by Euclidean lattices. Let $G = (V, E)$ be a multigraph such that a vertex is incident to an edge iff the corresponding polytope is a fundamental domain for the corresponding Euclidean lattice.

Definition 1.4. Let G be a multigraph and e be an edge of G connecting vertices v_0 and v_1 . Let L be the corresponding lattice of the edge e and X, Y be the corresponding polyhedra for v_0 and v_1 respectively. We translate every point in X to Y by vectors in L . Therefore, each loop based on the vertex v_0 in G corresponds to a *multigraph PET* on the polytope X .

1.3 Triple Lattice PETs

Let $G(V, E)$ be the graph with $V = \{A, B, C\}$ and $E = \{a_i, b_i, c_i\}_{i=1,2}$ as shown in Figure 1. The loop consisted of edges a_1, b_1, c_1 corresponds to a family of PETs called *triple lattice PETs*.

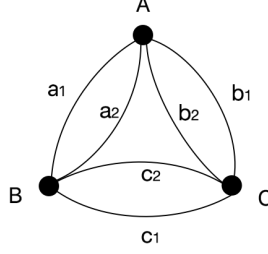


Figure 1: The multigraph graph G corresponding to the family of triple lattice PETs

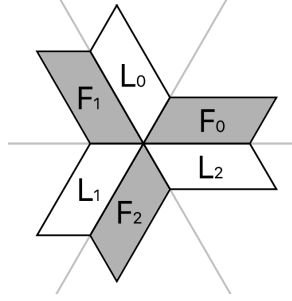


Figure 2: The scheme for triple lattice PETs

The construction of the triple lattice PETs are given as follows: Let F_0 be a parallelogram determined by the vectors $(2, 0)$ and $(s, \sqrt{3}s)$. Let r_0, r_1 and r_2 be the reflections about the lines in the directions of cubic roots of unity passing the origin. Let D_6 be the dihedral group of order 6 generated by r_0, r_1 and r_2 . The 6 parallelograms in Figure 2 are the orbit of F_0 under D_6 . Let F_j be the parallelogram as shown in Figure 2. Let L_j be the lattices generated by the sides of the parallelograms labeled by L_j . We will verify that F_j and $F_{j+1 \bmod 3}$ are fundamental domains for the lattice L_j in Section 2.

To define the triple lattice PETs, we consider the space $X'_s = \bigcup_{i=0}^2 F_i$. Given $p \in F_i$, define $g_s : X'_s \rightarrow X'_s$ by the formula:

$$g_s(p) = p + V_p \in F_{(i+1) \bmod 3}, \quad V_p \in L_i.$$

Definition 1.5. Let $X_s = F_0$. The triple lattice PET is defined by the map $f'_s : X_s \rightarrow X_s$ such that

$$f'_s = (g_s)^3.$$

For convenience, we translate X_s centered at the origin and keep all the translation vectors same. We denote the system by (X_s, f'_s) .

The figures below show an example of the triple lattice PET f_ϕ where ϕ is the golden ratio $\frac{\sqrt{5}-1}{2}$. The map f_ϕ is determined by partitions \mathcal{A} (left) and \mathcal{B} (right) such that each element in \mathcal{A} labeled by an integer i translates to an element in \mathcal{B} of the same number.

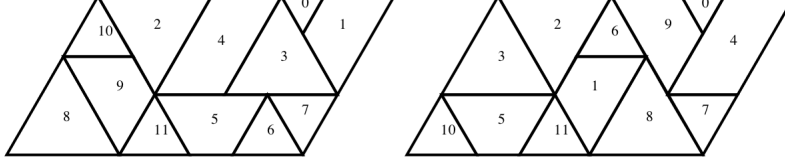


Figure 3: An example of triple lattice PET $f'_\phi : X_\phi \rightarrow X_\phi$, for $\phi = (\sqrt{5} - 1)/2$

1.4 Periodic Tiling and Limit Set

Suppose a point $p \in X_s$ is periodic under the triple lattice map f'_s . A periodic tile \diamond_p of f'_s is a maximal subset containing p such that f'_s is entirely defined and all points in \diamond_p have the same period as p . With computer assists, we can generate all periodic tiles in X_s . A periodic tiling Δ_s is the union of all periodic tiles \diamond_p for $p \in X_s$. Examples for periodic tiling are provided in Figure 4 for $s = \frac{\sqrt{5}-1}{2}$ and in Figure 6 for $s = 5/8$ and $8/13$. We will discuss the periodic tilings in Section 2 with more details provided.

A *limit set* of a periodic tiling Δ_s is a set of all points p such that every neighborhood of p intersects infinitely many tiles of Δ_s . Denote the limit set by Λ_s . An *aperiodic set* $\Gamma_s \subset \Lambda_s$ are the set of points with well-defined aperiodic orbits.

In order to have nicer limit sets, we apply a cut-and-paste operation on X_s which is illustrated in the figures below. The shaded region of X_s in Figure 4 is translated to the right. A precise description of cut-and-paste operation will be given in Section 2. Thus, we obtain the modified partitions \mathcal{A}' and \mathcal{B}' on X_s which produce a new family of PETs $f_s : X_s \rightarrow X_s$. Denote the system by (X_s, f_s) .

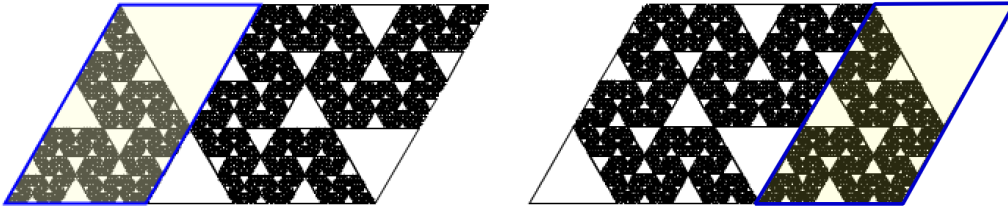


Figure 4: The figure on the left shows the limit set Λ_ϕ for $\phi = (\sqrt{5} - 1)/2$ in black. The figure on the right shows limit set Λ_ϕ after cut-and -paste operation (black).

1.5 Main Results

Define the *renormalization map* $R : (0, 1) \rightarrow [0, 1)$ to be the Gauss map

$$R(x) = \frac{1}{x} - \left\lfloor \frac{1}{x} \right\rfloor.$$

Note that the golden ration ϕ is a fixed point under the renormalization map. Let $Y_s \subset X$ be the parallelogram such that the left edge of Y_s is the left side of X_s and the bottom edge Y_s lies on the bottom of X_s of length

$$a = 2 - 2s \lfloor \frac{1}{s} \rfloor.$$

Theorem 1.6 (Renormalization). *Let $s \in [8/13, 13/21]$ and $t = R(s) \in [8/13, 5/8]$. The first return map $f_s|_{Y_s} : Y_s \rightarrow Y_s$ satisfies*

$$f_s|_{Y_s} = \psi_s^{-1} \circ f_t \circ \psi_s.$$

where ψ_s is a similarity which maps Y_s to X_t .

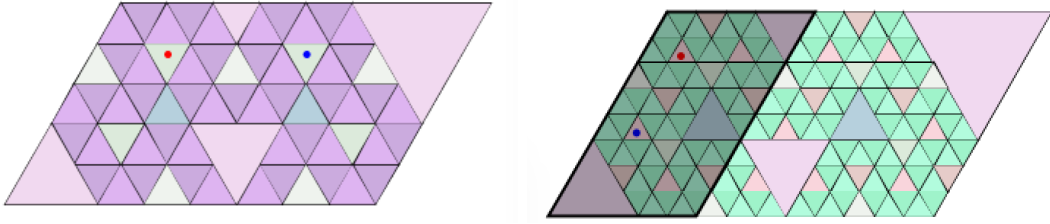


Figure 5: The figure on the left is an illustration of f_t for $t = 5/8 = R(8/13)$. Given a point (red) in X_t , its image is shown blue. The figure on the right shows Y_s (lightly shaded) for $s = 8/13$. The image of the given point (red) under the first return $f_s|_{Y_s}$ is shown in blue. The first return map $f|_Y$ is conjugate to f by a similarity.

Using renormalization theorem, the limit set Λ_ϕ for $\phi = \frac{\sqrt{5}-1}{2}$ can be obtained by applying a sequence of substitutions on isosceles trapezoids A, B and C in X_ϕ as shown Figure 6,7,8 and 9.

We deduce the following corollaries on the limit set Λ_ϕ :

Theorem 1.7. *The limit set Λ_ϕ is the limit of embedded polygons in \mathbb{R}^2 .*

Theorem 1.8. *The limit set Λ_ϕ has Lebesgue measure zero.*

Theorem 1.9. *The Hausdorff dimension of the limit set Λ_ϕ satisfies the property:*

$$\dim_H(\Lambda_\phi) = \frac{\log(-1 + \sqrt{2})}{\log \phi} = 1.83147 \dots < 2.$$

Acknowledgement

The author would like to thank her advisor Professor Richard Schwartz for his constant support, encouragement and guidance throughout this project. The author would also like to thank Yuhan Wang for helpful suggestions at different steps of the paper.

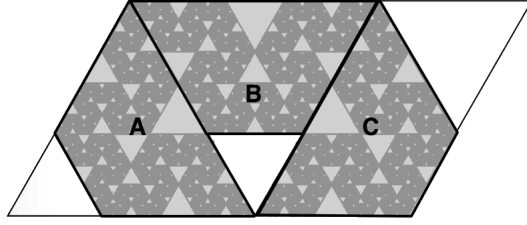


Figure 6: Isosceles trapezoids A, B, C in X_ϕ

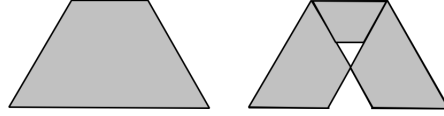


Figure 7: Substitution rule of the isosceles trapezoids

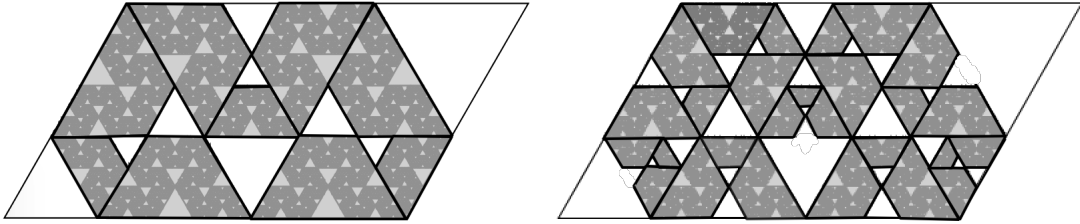


Figure 8: The iterations of the substitution rule on trapezoids A, B and C

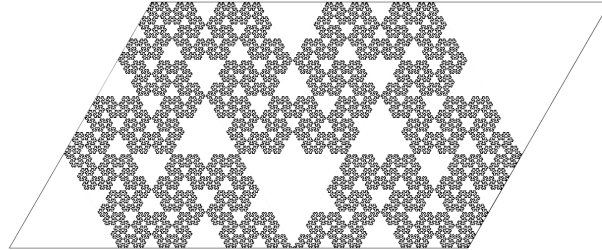


Figure 9: The limit set Λ_ϕ

2 Preliminaries

2.1 Fundamental Domains

We check an easy fact that each parallelogram F_i is a fundamental domain of L_i and $L_{(i-1) \bmod 3}$ for $i \in \{0, 1, 2\}$. This is because of two facts:

1. Each parallelogram F_i can tile the plane \mathbb{R}^2 via translation along the vectors in L_i and $L_{(i-1) \bmod 3}$. It implies that for every point $p \in \mathbb{R}^2$ can be translated into F_i by vectors in lattice L_i and $L_{(i-1) \bmod 3}$.

2.

$$\text{Area}(\mathbb{R}^2/L_j) = \text{Area}(F_i) = 2\sqrt{3}s.$$

for $i, j \in \{0, 1, 2\}$.

2.2 Cut-and-Paste Operation

Let \mathcal{A}, \mathcal{B} be two partitions which determines a triple lattice PET $f'_s : X_s \rightarrow X_s$. Each element A_i in the partition \mathcal{A} is either a subset of Y_s or $\text{Int}(A_i) \cap Y_s = \emptyset$. If $A_i \subset Y_s$, we translate A_i by the vector $(2 - a, 0)$. Else, we translate A_i by a vector $(-a, 0)$. Note that X_s is still centered at the origin. We apply the same procedure for elements in the partition \mathcal{B} . We obtain the modified partitions $\mathcal{A}', \mathcal{B}'$ on X_s which produce a new family of PETs $f_s : X_s \rightarrow X_s$.

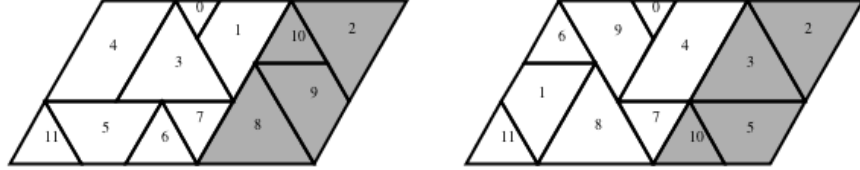


Figure 10: An illustration of the map $f_\phi : X_\phi \rightarrow X_\phi$ determined by two partitions $\mathcal{A}', \mathcal{B}'$ after applying a cut-and-paste operation on the partitions \mathcal{A} and \mathcal{B} as shown in Figure 3. The translation images of the sets $A \in \mathcal{A}$ and $B \in \mathcal{B}$ inside Y_s are shaded in gray.

Remark 2.1. Let $\{a_n\}_{n=0}^\infty$ be the sequence of Fibonacci numbers. If $s_n = \frac{a_{n-1}}{a_n}$, then the side lengths of Y_{s_n} are $2s_n$ and $2s_{n-1}$.

2.3 The Periodic Tiles

Let $\mathcal{A}_0 = \{A_i\}_{i=1}^m$ and $\mathcal{B}_0 = \{B_i\}_{i=1}^m$ be the partitions of polytopes such that the map f_s is determined by \mathcal{A} and \mathcal{B} . For integers $n \geq 2$, we inductively define \mathcal{A}_n to be the collection of polyhedra

$$f^{-n}(f^n(P) \cap A), \quad P \in \mathcal{A}_{n-1} \text{ and } A \in \mathcal{A}_0.$$

The partition \mathcal{A}_n is a refinement of \mathcal{A}_{n-1} . For each $n \geq 2$, the iteration f^n is not defined on

$$\bigcup_{P \in \mathcal{A}_n} \partial P.$$

Let P be an open polytope of \mathcal{A}_n . Every point in P must follow the same pattern of visits the elements in \mathcal{A}_0 . If a point $p \in P$ is periodic of period n , then all points of P are periodic with period n and P is a periodic tile defined in Section 1.4.

2.4 The Fiber Bundle Picture

In order to prove our main renormalization theorem, we reduce the calculations to convex polyhedra. The idea is inherited from [11] in Chapter 26. Here is the construction. We define

$$\mathcal{X} = \{(x, y, s) | (x, y) \in X_s, s \in [\frac{8}{13}, \frac{5}{8}]\}.$$

The space \mathcal{X} is a convex polyhedron and a fiber bundle over $[8/13, 5/8]$ such that the fiber above s is the parallelogram X_s . Let $F : \mathcal{X} \rightarrow \mathcal{X}$ be the fiber bundle map given by the formula:

$$F(x, y, s) = (f_s(x, y), s).$$

It is easy to see that F is a piecewise affine map. It is because for each $(x, y) \in X_s$, the map f_s is in the format of

$$f_s(x, y) = (x + m_0s + m_1, y + (n_0s + n_1)\sqrt{3})$$

where m_i, n_i are integers for $i = 0, 1$. If we vary the point (x, y) and the parameter s in a small neighborhood, the integers m_i, n_i , $i = 1, 2$ will not change.

2.5 Maximal Domains

Definition 2.2. A *maximal domain* of \mathcal{X} is a maximal subset in \mathcal{X} where F is entirely defined and continuous.

In other words, a maximal domain D of \mathcal{X} is a maximal subset of \mathcal{X} such that for all $(x, y, s) \in \text{Int}(D)$, the fiber bundle map $F : \mathcal{X} \rightarrow \mathcal{X}$ is continuous and in the form of

$$F(x, y, s) = (x, y, s) + (m_0s + m_1, y + (n_0s + n_1)\sqrt{3}, 0)$$

where the integers m_i, n_i are fixed for $i = 0, 1$. For each maximal domain D , there is a 4-tuple (m_0, m_1, n_0, n_1) associated to it. We call the tuple (m_0, m_1, n_0, n_1) the *coefficient tuple* of D .

By computer, we find that the polyhedron \mathcal{X} has a partition $\mathcal{D} = \{D_i\}_{i=0}^{11}$ of maximal domains D_i whose vertices can be written in the form of

$$(Az + B, (Cz + D)\sqrt{3}, z).$$

The coefficients A, B, C and D are integral multiples of $1/2$ and the value z is an end point of the interval $[8/13, 5/8]$ in our case. Each maximal domain has $2k$ of vertices where k of them have z -coordinate $8/13$ and another k have z -coordinate $5/8$. All vertices in the plane $z = 8/13$ pair with the vertices in $z = 5/8$ such that the constants A, B, C, D are same.

Let D_i be a maximal domain of \mathcal{X} with $2k$ vertices and coefficient tuple (m_0, m_1, n_0, n_1) . We represent D_i together with the coefficient tuple (m_0, m_1, n_0, n_1) by a $(k + 1) \times 4$ matrix shown as below:

$$\begin{pmatrix} A_1 & B_1 & C_1 & D_1 \\ \vdots & & & \\ A_k & B_k & C_k & D_k \\ m_0 & m_1 & n_0 & n_1 \end{pmatrix}.$$

We call such matrix a maximal domain matrix of \mathcal{X} under the map F . We provide the list of maximal domain matrix DT_i for $i = 0, 1, \dots, 11$ in Section 7.

2.6 Hausdorff dimension

Definition 2.3. Suppose that F is a compact set of \mathbb{R}^n and $s \geq 0$. Define

$$H_\delta^s(F) = \inf \left\{ \sum_{i=1}^{\infty} \text{diam}(U_i)^s \mid \text{diam}(U_i) \leq \delta \quad \text{and} \quad F \subset \bigcup_{i=1}^{\infty} U_i \right\}.$$

The Hausdorff measure is defined to be the limit

$$H^s(F) = \lim_{\delta \rightarrow 0} H_\delta^s(F).$$

There is a critical value s_0 of s at which H^s jumps from ∞ to 0. The value s_0 is called *Hausdorff dimension*. Formally, we have the definition

Definition 2.4.

$$\dim_H F = \inf\{s \geq 0 : H^s(F) = 0\} = \sup\{s : H^s(F) = \infty\}.$$

If $s = \dim_H F$, then $0 < H^s(F) < \infty$.

Here we state a classical result in [3] on Hausdorff dimensions of self-similarity sets. Let $\{\sigma_0, \sigma_1, \dots, \sigma_m\}$ be a set of similarities with ratios $0 < c_i < 1$ for $1 \leq i \leq m$. If the following conditions are satisfied:

1. attractor condition: the set F is an attractor i.e.

$$F = \bigcup_{i=1}^m \sigma_i(F),$$

2. open set condition: there exists non-empty bounded open set V such that

$$V \supset \bigsqcup_{i=1}^m \sigma_i(V),$$

then the Hausdorff dimension $\dim_H F = s$, where s is given by

$$\sum_{i=1}^m c_i^s = 1.$$

2.7 Computer Assistance

By computer, we give a proof for the main renormalization theorem and symmetries of periodic tilings (see Section 3 and 4). The proof involves calculations to determine if a given pair of polyhedra are nested or disjoint. All the calculations are done in integers or half integers, so there is no roundoff error. The pictures of partitions, periodic tilings and limit sets are taken from my java program. The program also do all the calculations. The program can be downloaded from the URL

https://www.math.brown.edu/~renyi/triple_lattice/project.html

3 Proof of the Renormalization Theorem

In this section, we want to show the existence of renormalization in the interval $[\frac{8}{13}, \frac{13}{21}]$. Suppose $s \in [8/13, 13/21]$ and $t = R(s)$. First, we give an informal description of the similarity $\psi_s : Y_s \rightarrow X_t$ in Theorem 1.6. Let (x_0, y_0) be the center of Y_s . We translate Y_s to center at the origin. Rotate the parallelogram (Y_s after translation) by $2\pi/3$ around the origin. Then, flip the obtained shape about the horizontal axis. Finally, multiply the side lengths of Y_s by $1/s$. Given $(x, y) \in Y_s$, an explicit formula of $\psi_s : Y_s \rightarrow X_t$ is given as follows:

$$x \mapsto \frac{1}{s} \left[(x - x_0) \cos\left(\frac{2\pi}{3}\right) - (y - y_0) \sin\left(\frac{2\pi}{3}\right) \right]$$

$$y \mapsto \frac{1}{s}[-(x - x_0) \sin(\frac{2\pi}{3}) + (y - y_0) \cos(\frac{2\pi}{3})].$$

Let \mathcal{Y} be the subset of \mathcal{X} defined as

$$\mathcal{Y} = \{(x, y, s) | (x, y \in Y_s), s \in [8/13, 13/21]\}.$$

Let $\Psi : \mathcal{X} \rightarrow \mathcal{X}$ be the map that piece together all the affine map ψ_s .

$$\Psi(x, y, s) = (\psi_s(x, y), R(s)).$$

The inverse map $\Psi^{-1} : \mathcal{X} \rightarrow \mathcal{X}$ is given by the formula:

$$\Psi^{-1}(x, y) = (\psi_s^{-1}(x, y), \frac{1}{1+s}).$$

Now we give the main calculation scheme of the proof. For each maximal domain D_i in \mathcal{X} of the fiber bundle map F , we check the following properties by computer:

1. There exists an integer $n \geq 0$ such that

$$F^n \circ \Psi^{-1}(D_i) \subseteq \mathcal{Y}$$

and

$$F^k \circ \Psi^{-1}(D_i) \cap \mathcal{Y} = \emptyset, \quad \text{for all } 0 \leq k < n.$$

2. $F^n \circ \Psi^{-1}(D_i) \subseteq \Psi^{-1} \circ F(D_i)$
3. Denote $F^n \circ \Psi^{-1}(D_i)$ by M_i .

$$\text{Int}(M_i) \cap \text{Int}(M_j) = \emptyset \quad \text{for } i \neq j.$$

- 4.

$$\sum_{i=0}^{11} \text{Volume}(M_i) = \text{Volume}(\mathcal{Y}).$$

The above computation shows that

$$F|_{\mathcal{Y}} = \Psi^{-1} \circ F \circ \Psi.$$

Therefore, the renormalization theorem on the interval $[8/13, 13/21]$ has been proved. \square

Definition 3.1. Let \mathcal{S} be a subset of \mathcal{X} and $F : \mathcal{X} \rightarrow \mathcal{X}$ be a piecewise affine map on \mathcal{X} . A maximal return domain of \mathcal{Z} is a maximal subset of the return map $F|_{\mathcal{Z}}$ where $F|_{\mathcal{Z}}$ is entirely defined and continuous.

It follows that the maximal return domains of \mathcal{Y} are in the form of $\Psi^{-1}(D_i)$ for $i = 0, \dots, 11$.

4 Structures of Periodic Tilings

Recall that Δ_s is the periodic tiling produced by the map $f_s : X_s \rightarrow X_s$. In this section, we analyze the symmetries and translation equivalences of Δ_s , which will help us to understand the limit set later.

4.1 Symmetry 1

Let L_s , L'_s , M_s and N_s be subsets of the parallelogram X_s as shown in Figure 10 and 11. In this section, we show that the periodic tiles in L_s and the periodic tiles in M_s are same up to rotation around the top vertex of L_s by $\pi/3$. Moreover, the periodic tiles in L'_s and the ones in N_s are same up to reflection about the vertical line passing through the bottom right vertex of L'_s .

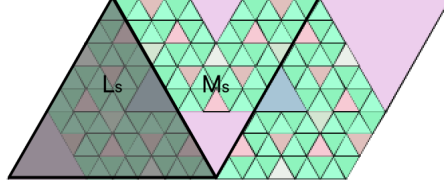


Figure 11: L_s (lightly shaded triangle) and M_s (transparent triangle) for $s = 8/13$

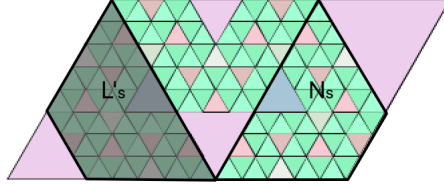


Figure 12: L'_s (lightly shaded trapezoid) and N_s (transparent trapezoid) for $s = 8/13$

Here is the precise description of the subsets. The subset $L_s \subset X_s$ is an equilateral triangle of side length $2s$. The top and left bottom vertices of L_s are the upper and lower left vertices of X_s . The lower right vertex of L_s lies on the bottom side of X_s . Let v_s be the rotation around the point $(\frac{s}{2} - 1, \frac{\sqrt{3}}{2}s)$ by the angle $\pi/3$. Define M_s as the image

$$M_s = v_s(L_s).$$

The set L'_s is an isosceles trapezoid given by the equation: $L'_s = L_s - T_s$, where $T_s \subset L_s$ is an equilateral triangle of side length $2s - 2(1 - \lfloor \frac{1}{s} \rfloor \cdot s)$. Let v'_s be the reflection about the vertical line $x = (3s - 1)/2$ which is a vertical line passing through the lower right vertex of L_s . The trapezoid N_s is the reflection image of L'_s

$$N_s = v'_s(L'_s).$$

The following lemma shows that there exists a rotational symmetry between periodic tiles in L_s and M_s , and a reflectional symmetry between tiles in L'_s and N_s .

Lemma 4.1. *Suppose $s \in [8/13, 13/21]$. Then, the following two equations are satisfied on M_s and N_s respectively:*

1.

$$v_s \circ f_s|_{L_s} \circ v_s^{-1} = f_s^{-1}|_{M_s}.$$

2.

$$v'_s \circ f_s|_{L'_s} \circ v'_s = f_s^{-1}|_{N_s}.$$

Proof. Define L to be a polyhedron given by

$$\mathcal{L} = \{(x, y, s) | (x, y) \in L_s \text{ and } s \in [8/13, 13/21]\}.$$

Define the polyhedra \mathcal{L}' , \mathcal{M} and \mathcal{N} in the same fashion.

We want to apply similar calculation to the one in Section 3.3. The difference is that we need to obtain the maximal return domain in \mathcal{M} of the return map $F^{-1}|_{\mathcal{M}}$. Let D_i be the maximal domain in \mathcal{X} with the coefficient tuple $T_i = (m_0, m_1, n_0, n_1)$ for $m_j, n_j \in \mathbb{Z}$ and $j = 0, 1$. Recall that the fiber bundle map F acts on the maximal domain D_i as the formula:

$$(x, y, s) \mapsto (x, y, s) + s(m_0, n_0, 0) + (m_1, n_1, 0)$$

for every point in $\text{Int}(D_i)$. For convenience, let $G = F^{-1}$ be the inverse map and $Q_i = F(D_i)$.

Here, we provide a pseudo-code to generate maximal return domains in \mathcal{M} . Let $M_i = Q_i \cap \mathcal{M}$.

1. Set $P = M_i$ and the tuple $T = (t_0, t_1, t_2, t_3) = -(m_0, m_1, n_0, n_1)$.
2. If $G(P) \cap \mathcal{M} \neq \emptyset$, then set $G(P) = G(P) \cap \mathcal{M}$. We obtain the maximal return domain P by translating each vertex (x, y, s) of the polyhedron $G(P)$ by the vector

$$-(t_0s + t_1, t_2s + t_3, 0).$$

The tuple $T = (t_0, t_1, t_2, t_3)$ is the coefficient tuple of P . Calculation terminates.

3. Set $G(P) = G(P) \cap \mathcal{M}^c$. There must exist some Q_j that is not a subset of \mathcal{M} such that $G(P) \cap Q_j \neq \emptyset$. For each Q_j satisfies this property, let $P = G(P) \cap Q_j$ and the tuple $T = (t_0, t_1, t_2, t_3) = (t_0 - m_0, t_1 - m_1, t_2 - n_0, t_3 - n_1)$ where (m_0, m_1, n_0, n_1) is the coefficient tuple of the maximal domain $D_j = G(Q_j)$.
4. If $G(P) \cap \mathcal{M} = \emptyset$, repeat step 2 and step 3.

By computer, the calculation terminates in finitely many steps. Therefore, we find all maximal return domains in \mathcal{M} of the map G . The polyhedron \mathcal{M} is partitioned into 12 maximal return domains. All maximal return domains along with their coefficient tuples will be provided in matrix format in Section 6.

Define the affine map $\Upsilon : \mathcal{X} \rightarrow \mathcal{X}$ as

$$\Upsilon(x, y, s) = (v_s(x, y), s)$$

The rest of the calculation is same as the calculation in the proof of the Renormalization Theorem (Section 3.3). The second statement is proved in the same way. All the maximal return domains in \mathcal{N} of the map G will be provided in matrix format in Section 6. \square

4.2 Symmetry 2

Let P_s be a isosceles trapezoid of the symmetric piece M_s with base angle $\pi/3$ such that P_s has vertices

$$\begin{bmatrix} \frac{1}{2}s - 1 \\ \frac{\sqrt{3}}{2}s \end{bmatrix} \begin{bmatrix} \frac{9}{2}s - 3 \\ \frac{\sqrt{3}}{2}s \end{bmatrix} \begin{bmatrix} \frac{3}{2}s - 1 \\ (\frac{7}{2}s - 2)\sqrt{3} \end{bmatrix} \begin{bmatrix} -\frac{s}{2} \\ (\frac{3}{2}s - 1)\sqrt{3} \end{bmatrix}.$$

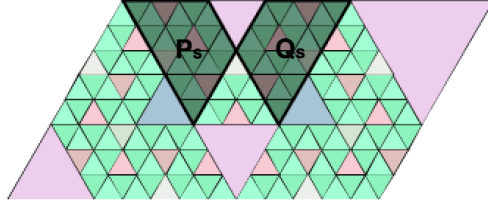


Figure 13: An illustration of reflectional symmetry of periodic tilings

Let $\iota_s : X_s \rightarrow X_s$ be the reflection about the vertical line $x = \frac{3}{2}s - 1$. Define the set Q_s as

$$Q_s = \iota(P_s).$$

The following lemma states that the periodic tiles in P_s and the ones in Q_s are same up to reflection ι_s .

Lemma 4.2. *Suppose $s \in [8/13, 13/21]$, then*

$$\iota_s \circ f_s^{-1}|_{P_s} \circ \iota_s = f_s|_{Q_s}.$$

We apply the same method as the proof of Lemma 4.1. Define \mathcal{Q} be the fiber bundle over $[8/13, 13/21]$

$$\mathcal{Q} = \{(x, y, s) | (x, y) \in Q_s \text{ and } s \in [8/13, 13/21]\}.$$

The list of maximal return domains in the polyhedra \mathcal{Q} along with their coefficient tuples are provided in Section 6.

4.3 Translation Equivalences

Suppose $s \in [8/13, 13/21]$. We first give a pictorial illustration to explain the structure of the periodic tiling Δ_s . Let \mathcal{P} be a partition of X_s as shown in Figure 14. The elements of \mathcal{P} in red are the fixed tiles of f_s . The shapes shaded in light or dark gray are translation equivalent to the shape of the same shade. The triangles labeled by A, B, C are same up to rotations and reflections.

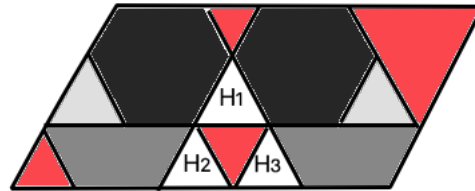


Figure 14: An illustration of the partition P of X_s

By analyzing the maximal domains D_i in \mathcal{X} and their coefficient tuples, we find that there always exist a partition of X_s as described above for $s \in [8/13, 13/21]$. Let us only focus on the sets shaded in gray. The parallelogram Y_s contains a semi-regular hexagon U_s , a triangle V_s and a trapezoid W_s . Explicit parametrizations of U_s, V_s, W_s are provided below:

- The vertices of U_s are

$$\begin{bmatrix} \frac{1}{2}s - 1 \\ \frac{\sqrt{3}}{2}s \end{bmatrix} \begin{bmatrix} \frac{9}{2}s - 3 \\ \frac{\sqrt{3}}{2}s \end{bmatrix} \begin{bmatrix} \frac{3}{2}s - 1 \\ (\frac{7}{2}s - 2)\sqrt{3} \end{bmatrix} \begin{bmatrix} -\frac{s}{2} \\ (\frac{3}{2}s - 1)\sqrt{3} \end{bmatrix} \begin{bmatrix} \frac{11}{2}s - 4 \\ (\frac{3}{2} - 1)\sqrt{3} \end{bmatrix} \begin{bmatrix} \frac{7}{2}s - 3 \\ (\frac{7}{2}s - 2)\sqrt{3} \end{bmatrix}$$

The translation vector on U_s of the map f_s is $(-2s + 2, 0)$. Let $U'_s = f_s(U_s)$. The pair of hexagons U_s and U'_s are illustrated as the dark-shaded hexagons in Figure 12.

- The vertices of V_s are

$$\begin{bmatrix} \frac{3}{2}s - 2 \\ (\frac{3}{2}s - 1)\sqrt{3} \end{bmatrix} \begin{bmatrix} \frac{11}{2}s - 4 \\ (\frac{3}{2}s - 1)\sqrt{3} \end{bmatrix} \begin{bmatrix} \frac{7}{2}s - 3 \\ (\frac{7}{2}s - 2)\sqrt{3} \end{bmatrix}.$$

The translation vector on V_s of the map f_s is $(-2s + 2, 0)$. Let $V' = f_s^2(U)$. The set V_s and V'_s are the light shaded triangles in Figure 12.

- The vertices of W_s are

$$\begin{bmatrix} \frac{7}{2}s - 3 \\ -\frac{\sqrt{3}}{2}s \end{bmatrix} \begin{bmatrix} -\frac{5}{2}s + 1 \\ -\frac{\sqrt{3}}{2}s \end{bmatrix} \begin{bmatrix} \frac{3}{2}s - 2 \\ (\frac{3}{2} - 1)\sqrt{3} \end{bmatrix} \begin{bmatrix} -\frac{s}{2} \\ (\frac{3}{2} - 1)\sqrt{3} \end{bmatrix}.$$

The translation vector on U_s of the map f_s is $(2s, 0)$. Let $W'_s = f_s(W_s)$. The set W_s and W'_s are the gray trapezoids in the figure above.

Let $Z_s = U_s \cup V_s \cup W_s$ and $Z'_s = U'_s \cup V'_s \cup W'_s$. It implies that the periodic tiles in Z_s are same as the ones in Z'_s . Let $s \in [8/13, 13/21]$ and $t = R(s)$. Since $Z_s \subset Y_s$ and the map $f|_{Y_s}$ is conjugate to f via a similarity, it follows that each non-fixed periodic tile $\diamond_p \in \Delta_t$ appears in the periodic tilings Δ_s twice up to similarities.

Now, we consider the triangles H_1, H_2 and H_3 as shown in Figure 14. Suppose $s \in [13/21, 21/34]$. Let $t = R(s)$ and $r = R^2(s)$. Denote the parallelogram $\psi_s^{-2}(X_s)$ by Z_s which is illustrated in Figure 13. Define the trapezoids

$$L'_r = \psi^{-2}(L_r), \quad M'_r = \psi^{-2}(M_r) \quad \text{and} \quad N'_r = \psi^{-2}(N_r)$$

where L_r, M_r and N_r are the sets of symmetry defined in Section 4.1.

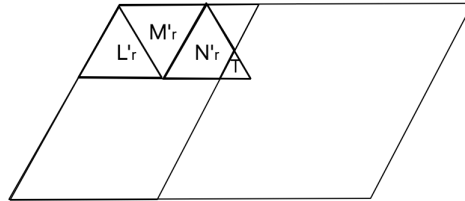


Figure 15: Sets of symmetry of Z_s

We claim that all the periodic tiles in the triangles H_1, H_2 and H_3 can be obtained from periodic tiles in L'_r, M'_r, N'_r and one extra periodic triangle illustrated in Figure 15. This is because

$$L'_r = f_s(H_2), \quad H_3 = f_s^2(N'_r) \cup f_s^2(T)$$

where T is a periodic tile in the shape of an equilateral triangle with side length $10s - 8$. The left edge of T is adjacent to the trapezoid N'_r as in Figure 15. Recall the map v_s and v'_s which is defined in Section 4.1. For the set H_1 , we have the relation

$$H_1 = f_s \circ v_s \circ v'_s(M'_r).$$

Let $u = R^s(s)$. It follows that that each periodic tile in triangle H_i in Figure 6 is same as a periodic tile in X_u up to similarity.

5 Limit Set Λ_ϕ for the Golden Ratio $\phi = \frac{\sqrt{5}-1}{2}$

5.1 The Fundamental Trapezoids

Let A, B and C be the isosceles trapezoid such that

$$A = L'_\phi, \quad B = v(A) \quad \text{and} \quad C = v'(A).$$

where L'_ϕ, v and v' are defined in Section 4. The sets A, B and C are shown in Figure 6. Let L_0 be the edge of A such that the intersection of A and B belongs to L_0 . We say that a tile *abuts* the the line segment L_0 if an edge of the tile is contained in L_0 . We call the segment the contacts between the periodic tile and L_0 .

Lemma 5.1. *There is a sequence of periodic triangles $\{P_n\}_{n=0}^\infty$ of Δ_ϕ satisfying the following properties:*

1. *Each P_n in the sequence abuts the segment L_0 .*
2. *The sequence occur in a monotone decreasing size. The periodic tiles P_n and P_{n+1} are similar triangles with scaling factor ϕ .*
3. *The periodic tiles in the sequence, from largest to smallest, move towards the point v .*
4. *For any point $p \in L_0 - v$, there must be a periodic tile P_n in the sequence whose contact of positive length contains p .*

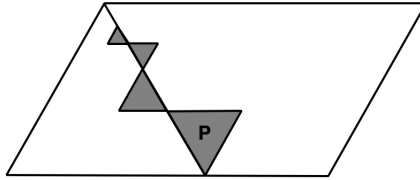


Figure 16: The sequence of periodic tiles in Lemma 6.1

Proof. Let P be the set as shown in Figure 15. The triangle P is a fixed periodic tile in Δ_ϕ . Recall that the map ψ_s is the similarity of scaling factor $1/s$ in Theorem 1.6. For convenience, we write ζ for the map ψ_ϕ^{-1} . The scaling factor of the similarity ζ is ϕ . The tile P abuts L_0 . By renormalization theorem, the set $\zeta(P)$ is a also a periodic tile in Δ_ϕ , and $\zeta(P)$ abuts the

segment L_0 . Moreover, the tile $\zeta(P)$ meet P at a vertex on L_0 , and the vertices of P and $\zeta(P)$ which do not belong to the line segment L_0 lie on the opposite sides of L_0 .

Therefore, by repeating the argument above, we obtain a sequence of periodic tiles $\{\zeta^n(P)\}_{n=0}^\infty$ which abuts L_0 . All points in the lemma follows directly from this structure. \square

Corollary 5.2. *Every point of L_0 , except for the top vertex of L_s , is contained in the edge of a periodic triangle given by the map f_ϕ .*

It is clear that $\Lambda_\phi \subset (A \cup B \cup C)$. Moreover, Corollary 6.2 implies that the only entrance for the limit set Λ_ϕ in A to get into the set B is at the vertex v . By symmetry, the the limit set in B can only enter the set C at the top right vertex of B . We call the trapezoids A, B and C the fundamental trapezoids.

5.2 Substitution Rule

We describe a substitution rule for fundamental trapezoids. The substitution is applied in the same fashion for all trapezoids up to similarity. For convenience, here we describe the substitution for an isosceles trapezoid M of base angle $\pi/3$ whose parallel edges are horizontal as shown in Figure 6 and 7. The trapezoid M is substituted by three similar isosceles trapezoids M_1, M_2 and M_3 with scaling factor ϕ, ϕ^2 and ϕ respectively. The trapezoids M_1 and M_3 are same up to reflection about the vertical line passing through the midpoint of the bottom of M . The longer parallel side of M_1 and M_3 are the non-parallel sides of M . The bottom of the trapezoid M_2 is the shorter parallel side of M .

We claim that the periodic tilings restricting in the substituted trapezoids M_1, M_2 and M_3 are same as the periodic tiling in M of Δ_ϕ up to similarities. Because of the symmetry, we focus the trapezoids A_1, A_2 and A_3 (see Figure 17) by substituting the fundamental trapezoid A .

- Write ψ as the similarity ψ_ϕ^{-1} defined in Theorem 1.6. The trapezoid A_1 is a symmetric piece in $\psi(X_\phi)$. Therefore, the periodic tiles in A and A_1 are same up to similarity.
- Note that $v(A_2) \subset H_1$ where v is the map of rotation discussed in Section 4.1. Moreover, the periodic tiles in A_2 are translation equivalent to the tiles in the set of symmetries $M'_r \subset \psi^2(X_\phi)$ in Figure 15. Therefore, the periodic tilings in A and A_2 are similar.
- By symmetry 2 in Section 4, the periodic tiles in A_1 and A_3 are similar up to reflection.

It follows that the periodic tilings in A_1, A_2 and A_3 are same to the tilings in A up to similarity. By renormalization theorem, we can also view the periodic tiling in $\bigcup_{i=1}^3 A_i$ as the periodic tiling in A with two periodic tiles removed. For notation, we denote the the set of three fundamental trapezoids by \mathcal{C}_0 . Define \mathcal{C}_n inductively as the set of trapezoids obtained by applying substitution rule on each trapezoid in \mathcal{C}_{n-1} for $n \geq 1$.

5.3 Proof of Theorem 1.7

Let $L\Lambda$ be the limit of the chains \mathcal{C}_n as $n \rightarrow \infty$. We show that

$$L\Lambda = \Lambda_\phi.$$

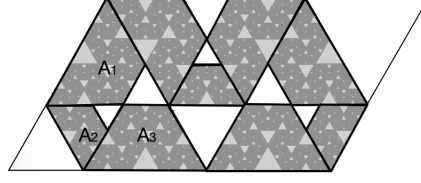


Figure 17: The substitution rule of a marked trapezoid

“ \subseteq ” Take an arbitrary point $p \in L\Lambda$. Since the sequence of chains $\{\mathcal{C}_n\}_0^\infty$ is a nested family, any neighborhood of p intersects an infinite sequence of marked trapezoids. By construction, each marked trapezoid contains infinitely many periodic tiles. Therefore, we have $p \in \Lambda_\phi$.

“ \supseteq ” Take a point $p \in \Lambda_\phi$, then every neighborhoods U_p of p intersects infinitely many periodic tiles. There must exists a trapezoids $M \in \mathcal{C}_{n_0}$ so that $p \in M$ and $M \cap U_p \neq \emptyset$. We want to show that p belongs to the chain \mathcal{C}_n for every $n \geq n_0$. Suppose there exists a marked trapezoid M_n such that $p \in M_n \setminus M'$ for any marked trapezoid $M' \subset M$ and $M' \in \mathcal{C}_{n+1}$, then we can find a neighborhood of p intersecting only finitely many periodic tiles in Δ_ϕ . Contradiction. \square

5.4 Proof of Theorem 1.8

Let $\mu(\mathcal{C}_n)$ be the Lebesgue measure of the union $\bigcup_{M \in \mathcal{C}_n} M$ of the isosceles trapezoids in \mathcal{C}_n .

According to the substitution, we have

$$\mu(\mathcal{C}_{n+1}) = (2\phi^2 + \phi^4)\mu(\mathcal{C}_n).$$

for $\phi = \frac{\sqrt{5}-1}{2}$. Since the scalar $2\phi^2 + \phi^4 = 0.90983 \dots < 1$, the Lebesgue measure of the limit set

$$\mu(\Lambda_\phi) = 0.$$

\square

5.5 Proof of Theorem 1.9

Let us focus on the substitutions in marked trapezoid A in the chain \mathcal{C}_0 first. Let σ_i be the similarity on the trapezoid A to smaller trapezoids A_i given by substitution for $i \in \{1, 2, 3\}$. By construction, the limit set Λ_ϕ satisfies the property

$$\Lambda_\phi = \bigcup_{i=1}^3 \sigma_i(\Lambda_\phi).$$

Moreover, to check the open set condition, we can take the open set V as the interior of A . Therefore, the open set condition holds. Thus, by Theorem 9.3 in [?], we have the equation:

$$2\phi^s + \phi^{2s} = 1.$$

The Hausdorff dimension for Λ_ϕ in the polygon A is $s = \frac{\log(-1+\sqrt{2})}{\log \phi} = 1.83147 \dots < 2$. Since we can apply the same computation on the limit set Λ_ϕ restricting in the trapezoids B and C . We have

$$1 < \dim_H(\Lambda_\phi) = \frac{\log(-1 + \sqrt{2})}{\log \phi} = 1.83147 \dots < 2. \quad \square$$

6 The Computational Data

Section 2.3

Here are the maximal domains for F in matrix format:

$$\begin{aligned}
 DT_0 &= \begin{pmatrix} -1/2 & -1 & -1/2 & 0 \\ 7/2 & -3 & -1/2 & 0 \\ 3/2 & -2 & 3/2 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix}, & DT_1 &= \begin{pmatrix} 1/2 & 0 & -3/2 & 1 \\ 9/2 & -2 & -3/2 & 1 \\ 5/2 & -1 & 1/2 & 0 \\ 1 & -1 & 1 & -1 \end{pmatrix}, \\
 DT_2 &= \begin{pmatrix} -1/2 & 1 & -1/2 & 0 \\ 3/2 & 0 & 3/2 & -1 \\ 1/2 & 0 & -3/2 & 1 \\ 9/2 & -2 & -3/2 & 1 \\ 0 & -1 & 2 & -1 \end{pmatrix}, & DT_3 &= \begin{pmatrix} 3/2 & -1 & -1/2 & 0 \\ -1/2 & 1 & -1/2 & 0 \\ 1/2 & 0 & -3/2 & 1 \\ 2 & -2 & 0 & 0 \end{pmatrix}, \\
 DT_4 &= \begin{pmatrix} 3/2 & -1 & -1/2 & 0 \\ -1/2 & 0 & 3/2 & -1 \\ 7/2 & -2 & 3/2 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix}, & DT_5 &= \begin{pmatrix} -5/2 & 1 & -1/2 & 0 \\ 3/2 & -1 & -1/2 & 0 \\ -1/2 & 0 & 3/2 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix}, \\
 DT_6 &= \begin{pmatrix} 7/2 & -3 & -1/2 & 0 \\ -5/2 & 1 & -1/2 & 0 \\ 3/2 & -2 & 3/2 & -1 \\ -1/2 & 0 & 3/2 & -1 \\ 2 & 0 & 0 & 0 \end{pmatrix}, & DT_7 &= \begin{pmatrix} 3/2 & -2 & 3/2 & -1 \\ 11/2 & -4 & 3/2 & -1 \\ 1/2 & -1 & 1/2 & 0 \\ 9/2 & -3 & 1/2 & 0 \\ -2 & 2 & 0 & 0 \end{pmatrix}, \\
 DT_8 &= \begin{pmatrix} 11/2 & -4 & 3/2 & -1 \\ 7/2 & -2 & 3/2 & -1 \\ 9/2 & -3 & 1/2 & 0 \\ -2 & 2 & 0 & 0 \end{pmatrix}, & DT_9 &= \begin{pmatrix} 3/2 & 0 & 3/2 & -1 \\ 5/2 & -1 & 1/2 & 0 \\ 1/2 & 1 & 1/2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \\
 DT_{10} &= \begin{pmatrix} 7/2 & -2 & 3/2 & -1 \\ 3/2 & -1 & 7/2 & -2 \\ -3/2 & 1 & 1/2 & 0 \\ 5/2 & -1 & 1/2 & 0 \\ 0 & -1 & -2 & 1 \end{pmatrix}, & DT_{11} &= \begin{pmatrix} 3/2 & -1 & 7/2 & -2 \\ 9/2 & -3 & 1/2 & 0 \\ -3/2 & 1 & 1/2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.
 \end{aligned}$$

Lemma 4.1

The maximal return domains in \mathcal{M} along with the coefficient tuple for $F|_{\mathcal{M}}$ in matrix format:

$$\begin{aligned}
 &\begin{pmatrix} 3/2 & -1 & -1/2 & 0 \\ -1/2 & 0 & 3/2 & -1 \\ 7/2 & -2 & 3/2 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix}, & &\begin{pmatrix} 1/2 & 0 & -3/2 & 1 \\ -3/2 & 1 & 1/2 & 0 \\ 5/2 & -1 & 1/2 & 0 \\ 2 & -2 & 0 & 0 \end{pmatrix} \\
 &\begin{pmatrix} -13/2 & 4 & 3/2 & -1 \\ 7/2 & -2 & 3/2 & -1 \\ -7/2 & 2 & -3/2 & 1 \\ 1/2 & 0 & -3/2 & 1 \\ 2 & -1 & 2 & -1 \end{pmatrix}, & &\begin{pmatrix} 13/2 & -4 & -3/2 & 1 \\ 1/2 & 0 & -3/2 & 1 \\ 7/2 & -2 & -9/2 & 3 \\ -4 & 2 & 0 & 0 \end{pmatrix} \\
 &\begin{pmatrix} 13/2 & -4 & -3/2 & 1 \\ 3/2 & -1 & 7/2 & -2 \\ 7/2 & -2 & -9/2 & 3 \\ -3/2 & 1 & 1/2 & 0 \\ -3 & 2 & 3 & -2 \end{pmatrix}, & &\begin{pmatrix} -7/2 & 2 & -3/2 & 1 \\ 13/2 & -4 & -3/2 & 1 \\ 3/2 & -1 & 7/2 & -2 \\ 0 & 0 & 0 & 0 \end{pmatrix} \\
 &\begin{pmatrix} 9/2 & -3 & 13/2 & -4 \\ 15/2 & -5 & 7/2 & -2 \\ 3/2 & -1 & 7/2 & -2 \\ -6 & 4 & 0 & 0 \end{pmatrix}, & &\begin{pmatrix} 3/2 & -1 & 7/2 & -2 \\ 9/2 & -3 & 1/2 & 0 \\ -3/2 & 1 & 1/2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \\
 &\begin{pmatrix} -1/2 & 0 & 3/2 & -1 \\ 9/2 & -3 & 13/2 & -4 \\ 5/2 & -2 & -3/2 & 2 \\ 15/2 & -5 & 7/2 & -2 \\ -3 & 2 & -3 & 2 \end{pmatrix}, & &\begin{pmatrix} -5/2 & 1 & 7/2 & -2 \\ 3/2 & -1 & 7/2 & -2 \\ -11/2 & 3 & 1/2 & 0 \\ 9/2 & -3 & 1/2 & 0 \\ 2 & -1 & -2 & 1 \end{pmatrix} \\
 &\begin{pmatrix} 5/2 & -2 & -3/2 & 1 \\ -5/2 & 1 & 7/2 & -2 \\ 15/2 & -5 & 7/2 & -2 \\ -2 & 2 & 0 & 0 \end{pmatrix}, & &\begin{pmatrix} -5/2 & 1 & 7/2 & -2 \\ 1/2 & -1 & 1/2 & 0 \\ -11/2 & 3 & 1/2 & 0 \\ 2 & -1 & -2 & 1 \end{pmatrix}.
 \end{aligned}$$

The maximal return domains in \mathcal{N} along with the coefficient tuple for $F|_{\mathcal{N}}$ in matrix format:

$$\begin{pmatrix} 3/2 & -1 & -1/2 & 0 \\ 11/2 & -3 & -1/2 & 0 \\ 7/2 & -2 & 3/2 & -1 \\ -1 & 1 & -1 & 1 \end{pmatrix}, \begin{pmatrix} 11/2 & -3 & -1/2 & 0 \\ -1/2 & 1 & -1/2 & 0 \\ 9/2 & -2 & 9/2 & -3 \\ 15/2 & -4 & 3/2 & -1 \\ -4 & 2 & 0 & 0 \end{pmatrix} \\
\begin{pmatrix} 11/2 & -3 & -1/2 & 0 \\ 17/2 & -5 & -7/2 & 2 \\ 5/2 & -1 & -7/2 & 2 \\ 2 & -1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 9/2 & -2 & 9/2 & -3 \\ 15/2 & -4 & 3/2 & -1 \\ 3/2 & 0 & 3/2 & -1 \\ -4 & 2 & 0 & 0 \end{pmatrix} \\
\begin{pmatrix} 17/2 & -5 & -7/2 & 2 \\ 5/2 & -1 & -7/2 & 2 \\ 7/2 & -2 & 3/2 & -1 \\ -5/2 & 2 & 3/2 & -1 \\ -3 & 2 & 3 & -2 \end{pmatrix}, \begin{pmatrix} 5/2 & -1 & -7/2 & 2 \\ -5/2 & 2 & 3/2 & -1 \\ 15/2 & -4 & 3/2 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \\
\begin{pmatrix} -5/2 & 2 & 3/2 & -1 \\ -17/2 & 6 & 3/2 & -1 \\ -11/2 & 4 & -3/2 & 1 \\ 3 & -2 & 3 & -2 \end{pmatrix}, \begin{pmatrix} 7/2 & -2 & 3/2 & -1 \\ -5/2 & 2 & 3/2 & -1 \\ 1/2 & 0 & -3/2 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \\
\begin{pmatrix} -17/2 & 6 & 3/2 & -1 \\ 3/2 & 0 & 3/2 & -1 \\ -11/2 & 4 & -3/2 & 1 \\ 9/2 & -2 & -3/2 & 1 \\ 6 & -4 & 0 & 0 \end{pmatrix}, \begin{pmatrix} -5/2 & 2 & 3/2 & -1 \\ 1/2 & -3/2 & 1 & -2 \\ 11/2 & -3 & 7/2 & -2 \\ -1/2 & 1 & 7/2 & -2 \\ 2 & -1 & -2 & 1 \end{pmatrix} \\
\begin{pmatrix} -11/2 & 4 & -3/2 & 1 \\ 9/2 & -2 & -3/2 & 1 \\ -1/2 & 1 & 7/2 & -2 \\ 1 & -1 & 1 & 1 \end{pmatrix}, \begin{pmatrix} 11/2 & -3 & 7/2 & -2 \\ -1/2 & 1 & 7/2 & -2 \\ 5/2 & 1 & 1/2 & 0 \\ 5 & -3 & -2 & 1 \end{pmatrix}.
\end{pmatrix}$$

The maximal return domains in \mathcal{Q} along with the coefficient tuple for $F|_{\mathcal{Q}}$ in matrix format:

$$\begin{pmatrix} 7/2 & -2 & -3/2 & -1 \\ -3/2 & 1 & 13/2 & -4 \\ 1/2 & 0 & -3/2 & 1 \\ -9/2 & 3 & 7/2 & -2 \\ 3 & -2 & -3 & 2 \end{pmatrix}, \begin{pmatrix} -3/2 & 1 & 13/2 & -4 \\ 13/2 & -4 & -3/2 & 1 \\ -19/2 & 6 & -3/2 & 1 \\ 2 & -1 & 2 & -1 \end{pmatrix} \\
\begin{pmatrix} 13/2 & -4 & -3/2 & 1 \\ -19/2 & 6 & -3/2 & 1 \\ 3/2 & -1 & 7/2 & -2 \\ -29/2 & 9 & 7/2 & -2 \\ 10 & -6 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 1/2 & 0 & -3/2 & 1 \\ -9/2 & 3 & 7/2 & -2 \\ 11/2 & -3 & 7/2 & -2 \\ 0 & 0 & 0 & 0 \end{pmatrix} \\
\begin{pmatrix} -19/2 & 6 & -3/2 & 1 \\ -29/2 & 9 & 7/2 & -2 \\ -9/2 & 3 & 7/2 & -2 \\ 5 & -3 & 5 & -3 \end{pmatrix}, \begin{pmatrix} 23/2 & -7 & 7/2 & -2 \\ -9/2 & 3 & 7/2 & -2 \\ 7/2 & -2 & -9/2 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix} \\
\begin{pmatrix} -9/2 & 3 & 7/2 & -2 \\ 11/2 & -3 & 7/2 & -2 \\ 7/2 & -2 & -9/2 & 3 \\ 27/2 & -8 & -9/2 & 3 \\ -5 & 3 & 5 & -3 \end{pmatrix}, \begin{pmatrix} 3/2 & -1 & 7/2 & -2 \\ 23/2 & -7 & 7/2 & -2 \\ -3/2 & 1 & 1/2 & 0 \\ 17/2 & -5 & 1/2 & 0 \\ -6 & 4 & 0 & 0 \end{pmatrix} \\
\begin{pmatrix} 7/2 & -2 & -9/2 & 3 \\ 27/2 & -8 & -9/2 & 3 \\ 17/2 & -5 & 1/2 & 0 \\ 3 & -2 & 3 & -2 \end{pmatrix}, \begin{pmatrix} 11/2 & -3 & 7/2 & -2 \\ 17/2 & -5 & 1/2 & 0 \\ 5/2 & -1 & 1/2 & 0 \\ -2 & 1 & -2 & 1 \end{pmatrix}.
\end{pmatrix}$$

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